Yau 2025 Applied Math – Team Contest

1. Given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ with piecewise hyperplanar boundary $\partial\Omega$. For $p \geq 1$, let $\mathcal{A}_p := W_0^{1,p}(\Omega;\mathbb{R})$ denote the standard Sobolev space, and $V_h \subset \mathcal{A}_p$ denote the subspace of all piecewise affine finite element functions with respect to a shape-regular and quasi-uniform triangulation $\mathcal{T}_h$ parametrized by the mesh size h. Consider the minimization of an energy $E : \mathcal{A}_p \to \mathbb{R} \cup \{+\infty\}$:

$$\inf E(\mathcal{A}_p) := \inf_{v\in\mathcal{A}_p} E(v), \quad \text{where} \quad E(v) := \int_\Omega W(x, v(x), \nabla v(x))\, dx, \ \forall\, v \in \mathcal{A}_\infty\,,$$

associated with a continuous energy density function $W\colon \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, and its finite element approximation:

$$E_h := \inf_{v^h\in V_h} E(v^h).$$

   (a) Show that if W is continuous, then

$$\lim_{h\to 0} E_h = \inf_{v\in\mathcal{A}_\infty} E(v).$$

   (b) Find an example of W on $\Omega = (0,1)$ such that

$$\lim_{h\to 0} E_h \neq \inf_{v\in\mathcal{A}_1} E(v).$$

2. The softmax function $\boldsymbol{p} = \mathrm{softmax}(\boldsymbol{z})$ is defined as

$$p_i = \frac{e^{z_i}}{\sum_j e^{z_j}}, \quad \text{for } \boldsymbol{z} \in \mathbb{R}^K.$$

Let $\mathcal{S}^{K-1} := \left\{\boldsymbol{p} \in \mathbb{R}^K \,\middle|\, p_i \geq 0, \sum_i p_i = 1\right\}$ be the probability simplex.

   (a) Define the log-sum-exp function

$$\mathrm{lse}(\boldsymbol{z}) := \log\left(\sum_{i=1}^K e^{z_i}\right).$$

   Prove that $\mathrm{lse}(\boldsymbol{z})$ is convex.

   (b) Show that the convex conjugate (Fenchel-Legendre transform) is:

$$\mathrm{lse}^*(\boldsymbol{y}) = \begin{cases} \sum_{i=1}^K y_i \log y_i & \text{if } \boldsymbol{y} \in \mathcal{S}^{K-1}, \\ \infty & \text{otherwise.} \end{cases}$$

   (c) Derive the biconjugate lse^{**} and verify that it equals lse.

   (d) Prove that

$$\mathrm{softmax}(\boldsymbol{z}) = \operatorname*{arg\,max}_{\boldsymbol{p}\in\mathcal{S}^{K-1}} \left(\boldsymbol{z}\cdot\boldsymbol{p} - \boldsymbol{p}\cdot\log\boldsymbol{p}\right).$$

3. Let $A = \lambda I + N \in \mathbb{C}^{n\times n}$, where N is nilpotent of index m, i.e., $N^m = 0$, and $\lambda \in \mathbb{C}$. Let f be analytic near λ.

   (a) Derive a general expression for $f(A)$ in terms of the derivatives of f at λ and powers of N.

   (b) Specialize to the case $f(z) = e^z$, and compute e^A explicitly.

   (c) If $A \in \mathbb{R}^{n\times n}$ is real but has complex eigenvalues, explain how to compute $e^{At} \in \mathbb{R}^{n\times n}$ efficiently.

   (d) Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Compute e^{At} and interpret the result.